Institute of Media, Information, and Network

Chapter 2 The Discrete-Time Fourier Transform

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Topic

- ☐ **2.1 Discrete-Time Fourier Transform (DTFT)**
- ☐ **2.2 Properties of the DTFT**
- ☐ **2.3 Convolution Property of the DTFT**
- ☐ **2.4 Multiplication Property of the DTFT**

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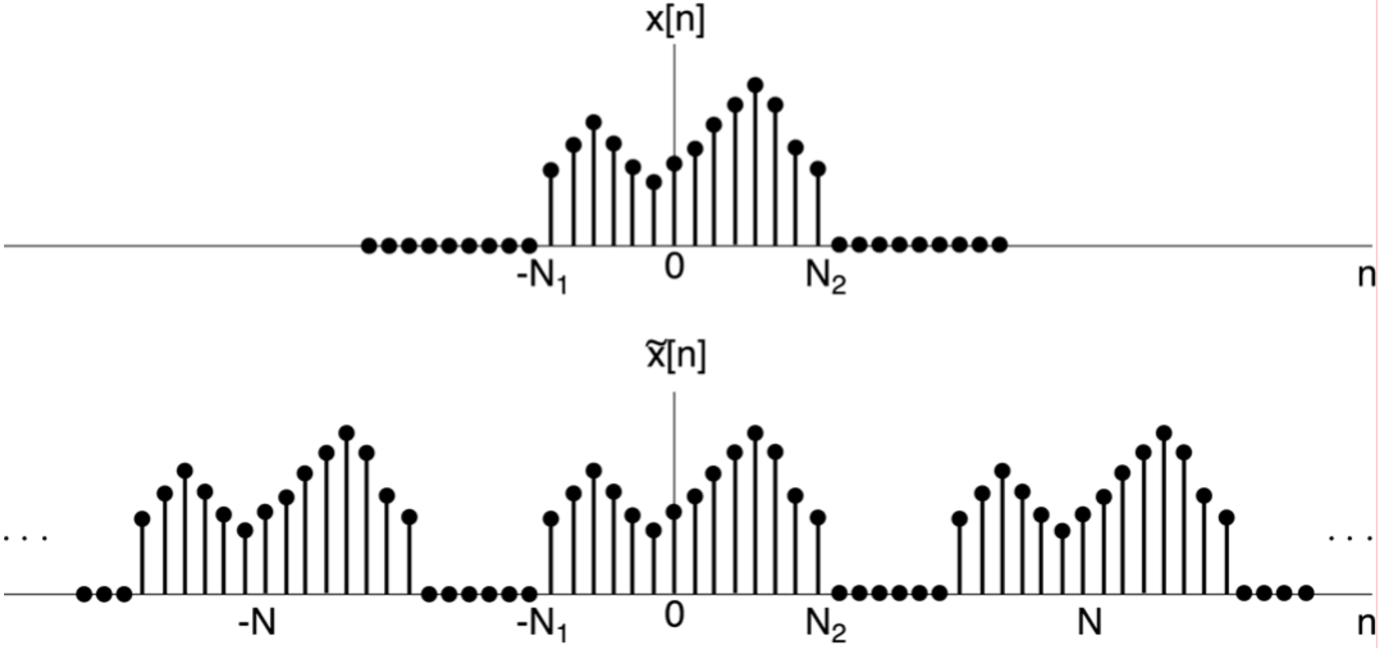
2.1.1 The Discrete-Time Fourier Transform

Derivation: (Analogous to CTFT) except $e^{jwn} = e^{j(w+2\pi)n}$)

- x[n] – aperiodic and (for simplicity) of finite duration
- N is large enough so that x[n] = 0 if |n| ≥ N/2
- x[n] = x[n] for [n] ≤ N/2 and periodic with period N



$$\tilde{x}[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \ \omega_0 = \frac{2\pi}{N}$$

DTFS synthesis eq.

$$a_k = \frac{1}{N} \sum_{n=<N>} \tilde{x}[n] e^{-jk\omega_0 n}$$

DTFS analysis eq.

$$= \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk\omega_0 n}$$

Define

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad \boxed{-\text{ periodic in } \omega \text{ with period } 2\pi}$$

$$\Downarrow$$

$$a_k = \frac{1}{N} X(e^{jk\omega_0})$$

$$\tilde{x}[n] = \sum_{k=<N>} \underbrace{\frac{1}{N} X(e^{jk\omega_0})}_{a_k} e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 \quad (*)$$

As $N \to \infty$: $\quad \tilde{x}[n] \to x[n]$ for every n

$$\omega_0 \to 0, \sum \omega_0 \to \int d\omega$$

The sum in $(*)$ $\to$ an integral

$\Downarrow$ The DTFT Pair

$$\boxed{\begin{array}{ll} x[n] = \dfrac{1}{2\pi} \displaystyle\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega & \text{Synthesis equation} \\[2em] X(e^{j\omega}) = \displaystyle\sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} & \text{Analysis equation} \end{array}}$$

Any 2π interval in ω

2.1.2 Convergence Issues

- Synthesis Equation: None, since integrating over a finite interval
- Analysis Equation: Need conditions analogous to CTFT, e.g.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$ — Finite energy

or

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$ — Absolutely summable

2.1.3 Examples of DT Fourier Transform

- Unit Impulse

$$x[n] = \delta[n] \leftrightarrow X(e^{j\omega}) = 1$$

- DC Signal

$$x[n] = 1 \leftrightarrow X(e^{j\omega}) = 2\pi \sum_{l=-\infty}^{\infty} \delta[\omega - 2\pi l]$$

Proof:

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}\,d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi \sum_{l} \delta(\omega - 2\pi l)e^{j\omega n}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\delta(\omega)e^{j\omega n}\,d\omega = 1$$

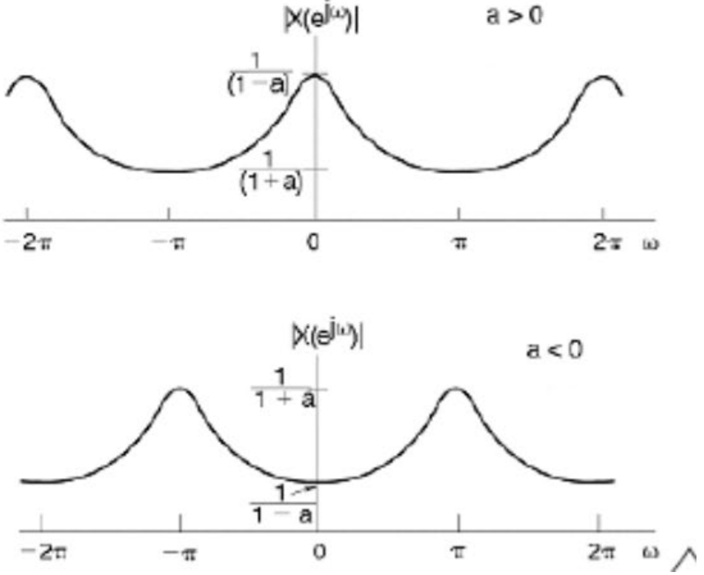
- Exponential Signal

$$x[n] = \alpha^n u[n] \leftrightarrow X(j\omega) = \frac{1}{1-\alpha e^{-j\omega}} \qquad |\alpha| < 1$$

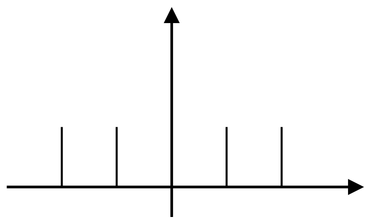
$$|X(e^{j\omega})| = \frac{1}{\sqrt{1-2a\cos\omega + a^2}}$$

$$\omega = 0:\ X(e^{j\omega}) = \frac{1}{\sqrt{1-2a+a^2}} = \frac{1}{1-a}$$

$$\omega = \pi:\ X(e^{j\omega}) = \frac{1}{\sqrt{1+2a+a^2}} = \frac{1}{1+a}$$



- Rectangular Pulse

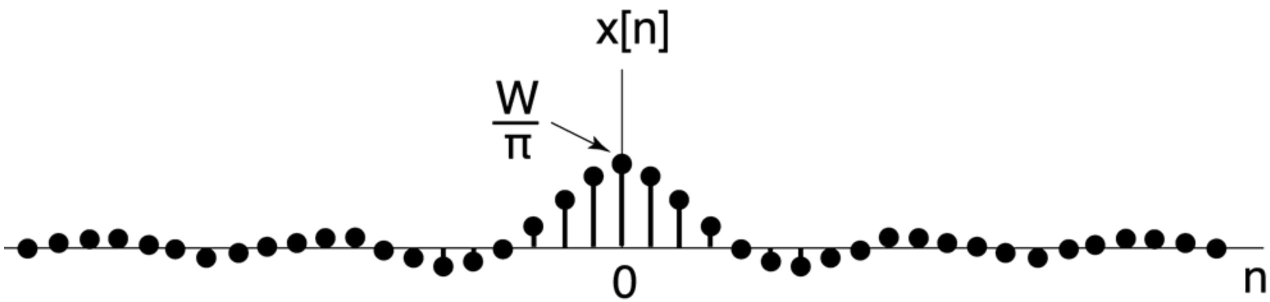


$$\leftrightarrow X(e^{j\omega}) = \frac{\sin \omega(N_1 + \frac{1}{2})}{\sin(\omega/2)} = X(e^{j(\omega-2\pi)})$$

Periodic

-

$$x(n) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n} \leftrightarrow X(e^{j\omega}) = \begin{cases} 1, & 0 \le |\omega| \le W \\ 0, & W < |\omega| \le \pi \end{cases}$$



Notes: $W < \pi$

2.1.4 DTFT of Periodic Signals

$$x[n] = x[n+N]$$

$$x[n] = \sum_{k=<N>} a_k e^{jk\omega_0 n}, \ \omega_0 = \frac{2\pi}{N}$$

DTFS synthesis eq.

From the last page: $e^{jk\omega_0 n} \longleftrightarrow 2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m)$

Linearity of DTFT

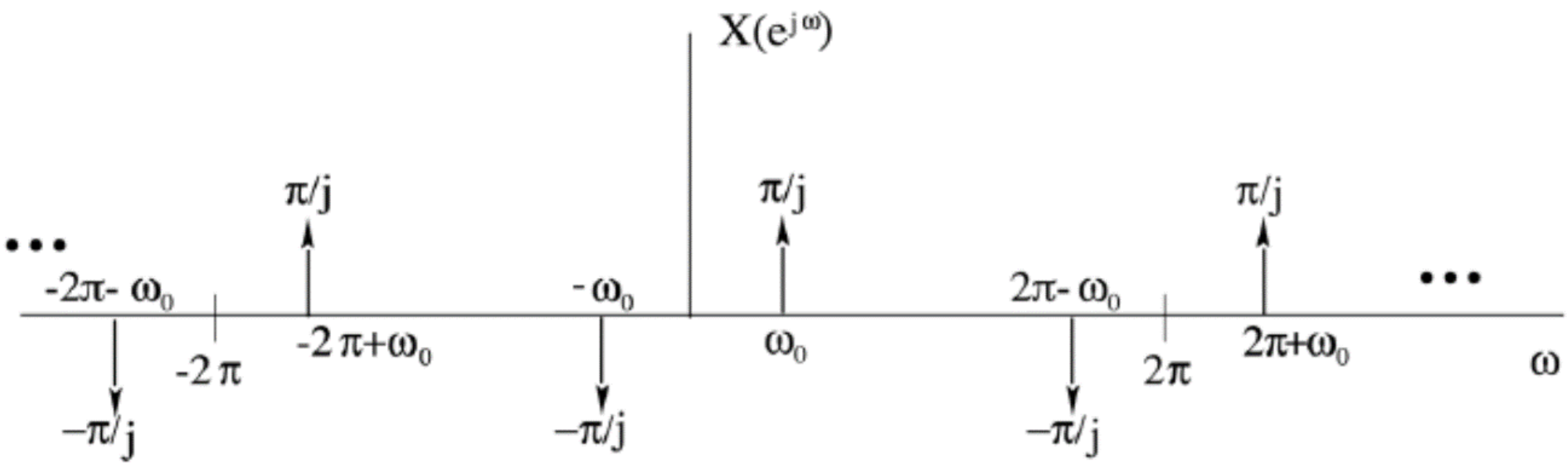
$$X(e^{j\omega}) = \sum_{k=<N>} a_k \left[2\pi \sum_{m=-\infty}^{\infty} \delta(\omega - k\omega_0 - 2\pi m) \right]$$

$$= 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - k\omega_0) = 2\pi \sum_{k=-\infty}^{\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

- DT sine function

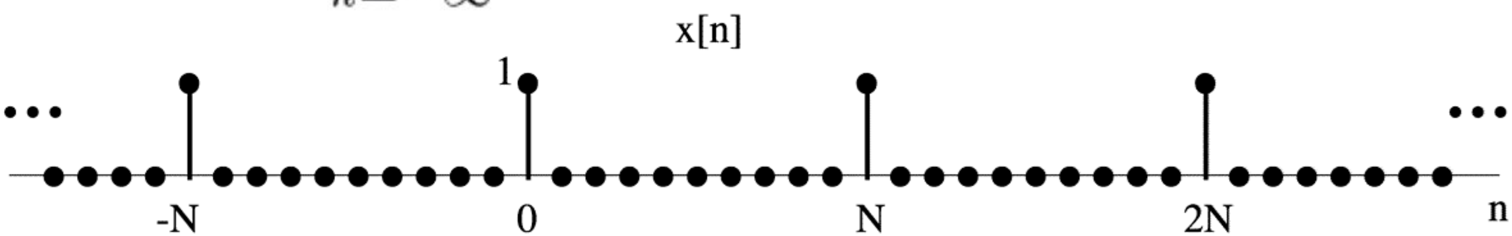
$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n}$$

$$X(e^{j\omega}) = \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi m) - \frac{\pi}{j} \sum_{m=-\infty}^{\infty} \delta(\omega + \omega_0 - 2\pi m)$$



- DT periodic impulse train

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \qquad \omega_0 = 2\pi/N$$

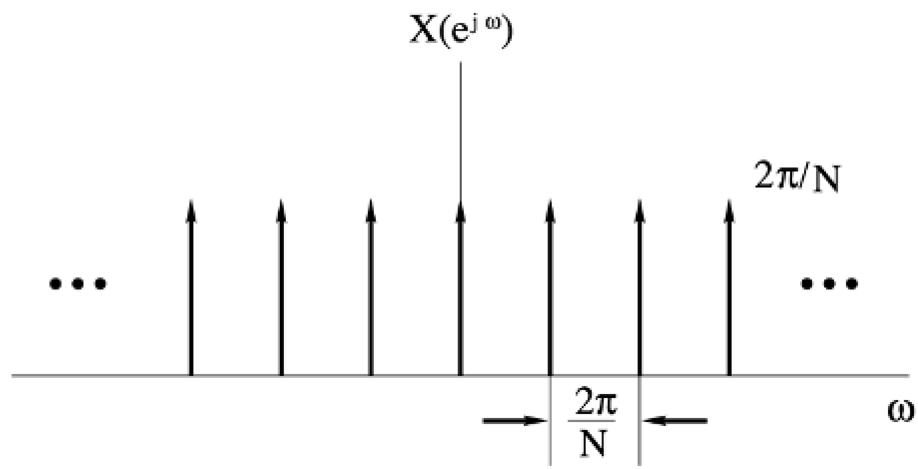


$$a_k = \frac{1}{N} \sum_{n=<N>} x[n] e^{-jk\omega_0 n}$$

$$= \frac{1}{N} \sum_{n=0}^{N-1} \underbrace{x[n]}_{=\delta[n]} e^{-jk\omega_0 n} = \frac{1}{N}$$

$$\Downarrow$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



— Also periodic impulse train – in the frequency domain!

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- 2.4 Multiplication Property of the DTFT

- Periodicity

$$\boxed{X(e^{j(\omega+2\pi)}) = X(e^{j\omega})}$$

— Different from CTFT

- Linearity

$$x_1[n] \leftrightarrow X_1(e^{j\omega}) \quad x_2[n] \leftrightarrow X_2(e^{j\omega})$$

$$\boxed{ax_1[n] + bx_2[n] \leftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})}$$

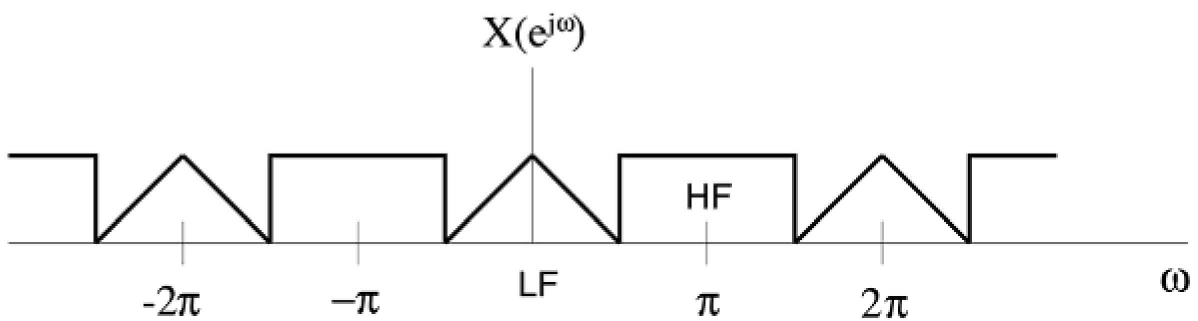
• Time Shifting and Frequency Shifting

$$x[n] \leftrightarrow X(e^{j\omega})$$

$$\boxed{\begin{aligned} x[n-n_0] &\leftrightarrow e^{-j\omega n_0} X(e^{j\omega}) \\ e^{j\omega_0 n} x[n] &\leftrightarrow X[e^{j(\omega-\omega_0)}] \end{aligned}}$$

— Important implications in DT because of periodicity

Example:



$$\omega_0 = \pi \, , \; y[n] = e^{j\pi n} x[n] = (-1)^n x[n]$$

- Time Reversal

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

- Conjugate Symmetry

$$x[n] \text{ real} \Rightarrow X(e^{j\omega}) = X^*(e^{-j\omega})$$

$$\Downarrow$$

$$|X(e^{j\omega})| \text{ and } \Re e\left\{X(e^{j\omega})\right\} \text{ are even functions}$$
$$\angle X(e^{j\omega}) \text{ and } \Im m\left\{X(e^{j\omega})\right\} \text{ are odd functions}$$

and

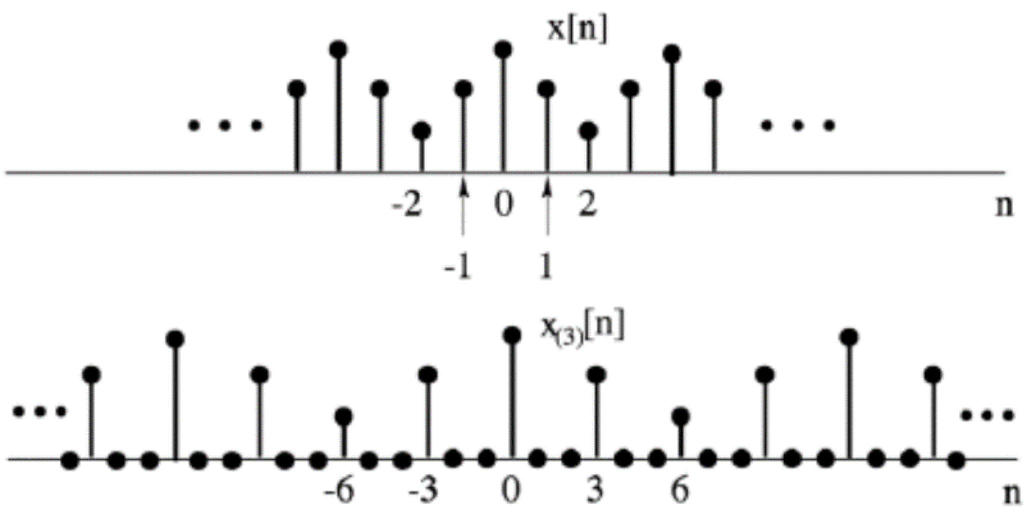
$$x[n] \text{ real and even} \Leftrightarrow X(e^{j\omega}) \text{ real and even}$$
$$x[n] \text{ real and odd} \Leftrightarrow X(e^{j\omega}) \text{ purely imaginary and odd}$$

- Time Expansion
  - Recall CT property: $x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\left(\frac{\omega}{a}\right)\right)$
  - But in DT: x[n/2] makes no sense

    x[2n] misses odd values of x[n]
  - We can "slow" a DT signal down by inserting zeros:

    k —an integer ≥ 1

    $x_{(k)}[n]$ — insert $(k - 1)$ zeros between successive values



Insert two zeros in this example (k=3)

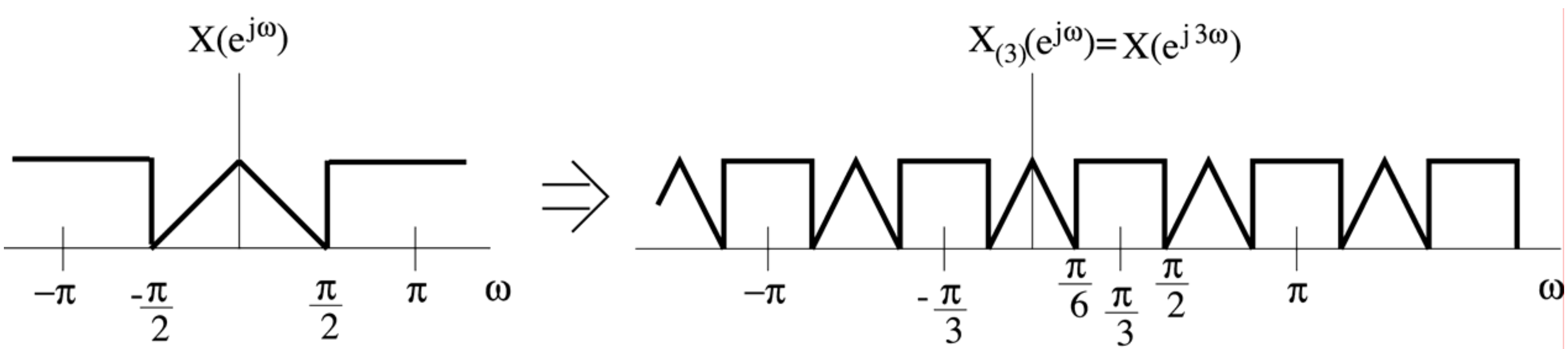
$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if n is an integer multiple of k} \\ 0 & \text{otherwise} \end{cases}$$

— Stretched by a factor of *k* in time domain

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_{(k)}[n]e^{-j\omega n} \overset{n=mk}{=} \sum_{m=-\infty}^{\infty} x_{(k)}[mk]e^{-j\omega mk}$$

$$= \sum_{m=-\infty}^{\infty} x[m]e^{-j(k\omega)m} = X(e^{jk\omega})$$

-compressed by a factor of *k* in frequency domain



- Differencing and Accumulation

$$x[n]-x[n-1]\leftrightarrow(1-e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^{n}x[m]\leftrightarrow\frac{1}{1-e^{-j\omega}}X(e^{j\omega})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

Example : $\because u[n]=\sum_{m=-\infty}^{n}\delta[m]$ $\quad \delta[n]\leftrightarrow 1$

$$\therefore u[n]\leftrightarrow\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)=\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

Notes: $\sum_{m=-\infty}^{n}x[m]=x[n]*u[n]$

- Differentiation in Frequency

$$\boxed{nx[n] \leftrightarrow j\frac{dX(e^{j\omega})}{d\omega}}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$\frac{d}{d\omega}X(e^{j\omega}) = -j\sum_{n=-\infty}^{\infty} nx[n]e^{-j\omega n}$$

$\Downarrow$ multiply by j on both sides

Multiplication by n

$$nx[n] \leftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$$

Differentiation in frequency

- Parseval’s Relation

$$\underbrace{\sum_{n=-\infty}^{\infty} |x[n]|^2}_{\text{Total energy in time domain}} = \underbrace{\frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega}_{\text{Total energy in frequency domain}}$$

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2.3.1 Convolution Property

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n] = h[n] * x[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

$\Rightarrow$ Frequency response $H(e^{j\omega})$ = DTFT of the unit sample response

- Example: Enginfunction property of discrete exponential

$$
\begin{aligned}
x[n] = e^{j\omega_0 n} \longleftrightarrow X(e^{j\omega}) &= 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
Y(e^{j\omega}) &= H(e^{j\omega}) 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
&= 2\pi \sum_{k=-\infty}^{\infty} H(e^{j(\omega_0 + 2\pi k)}) \delta(\omega - \omega_0 - 2\pi k) \\
&\overset{H \text{ Periodic}}{=} H(e^{j\omega_0}) 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi k) \\
&\Downarrow \\
y[n] &= H(e^{j\omega_0}) e^{j\omega_0 n}
\end{aligned}
$$

- Example: convolution of exponentials

$$h[n] = \alpha^n u[n], \quad x[n] = \beta^n u[n] \qquad |\alpha|, |\beta| < 1$$

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}$$

$$\Downarrow$$

$$y[n] = h[n] * x[n] \longleftrightarrow Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)\left(\frac{1}{1-\beta e^{-j\omega}}\right)$$

- ratio of polynomials in $e^{-j\omega}$

$$\beta \neq \alpha : \; Y(e^{j\omega}) \stackrel{\text{PFE}}{=} \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}$$

A, B - determined by partial fraction expansion

$$y[n] = A\alpha^n u[n] + B\beta^n u[n]$$

When α=β, by exploiting the differential property in frequency domain , one obtains

$$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2 \qquad \left(Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right) \right)$$

$$\alpha^n u[n] \xleftrightarrow{FT} \frac{1}{1-\alpha e^{-j\omega}}$$

$$n\alpha^n u[n] \xleftrightarrow{FT} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$(n+1)\alpha^{n+1} u[n+1] \xleftrightarrow{FT} je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$y[n] = (n+1)\alpha^n u[n]$$

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2.4.1 Multiplication Property of DTFT

$$
\begin{aligned}
y[n] = x_1[n] \cdot x_2[n] \longleftrightarrow Y(e^{j\omega}) &= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta \\
&= \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \\
& \hookrightarrow \text{ Periodic Convolution}
\end{aligned}
$$

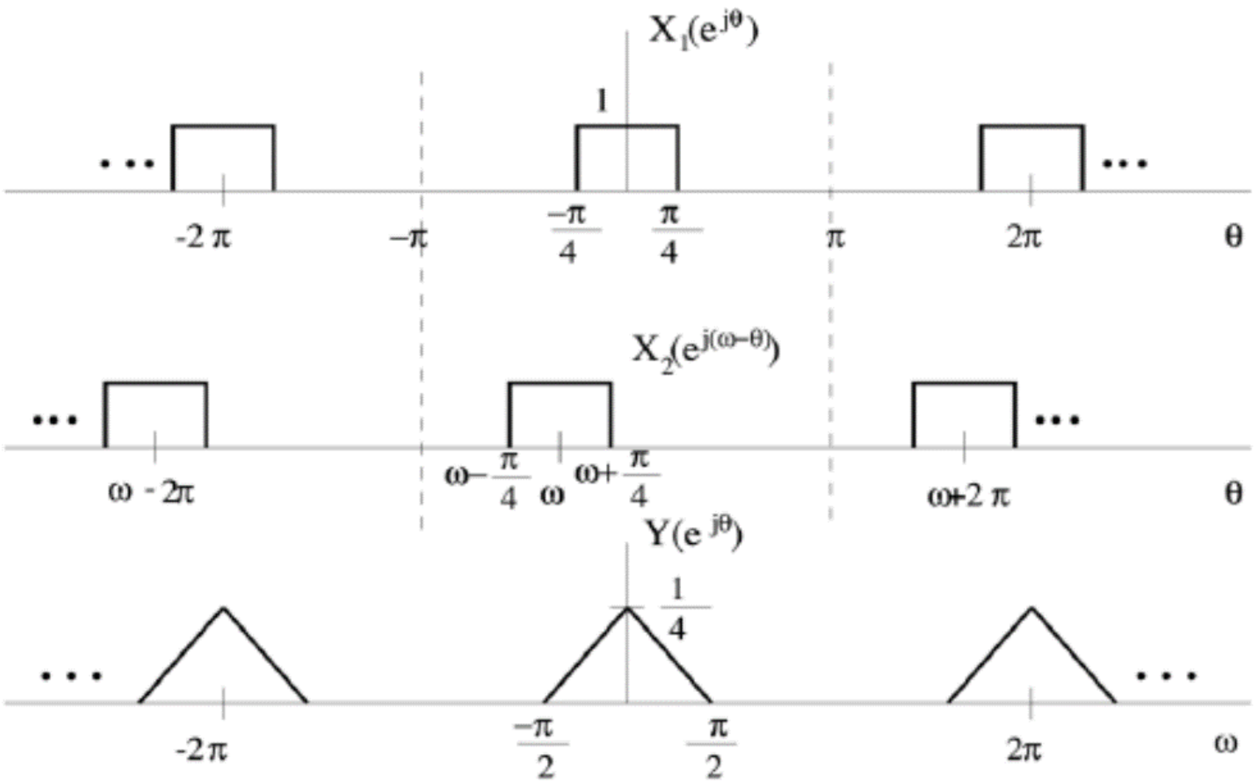
Derivation:

$$
\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x_1[n] \cdot x_2[n] e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) e^{j\theta n} d\theta \right) x_2[n] e^{-j\omega n} \\
&= \frac{1}{2\pi} \int_{2\pi} (X_1(e^{j\theta}) \underbrace{\sum_{n=-\infty}^{\infty} x_2[n] e^{-j(\omega-\theta)n}}_{X_2(e^{j(\omega-\theta)})}) d\theta \\
&= \frac{1}{2\pi} \int_{2\pi} X_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta
\end{aligned}
$$

- Example:

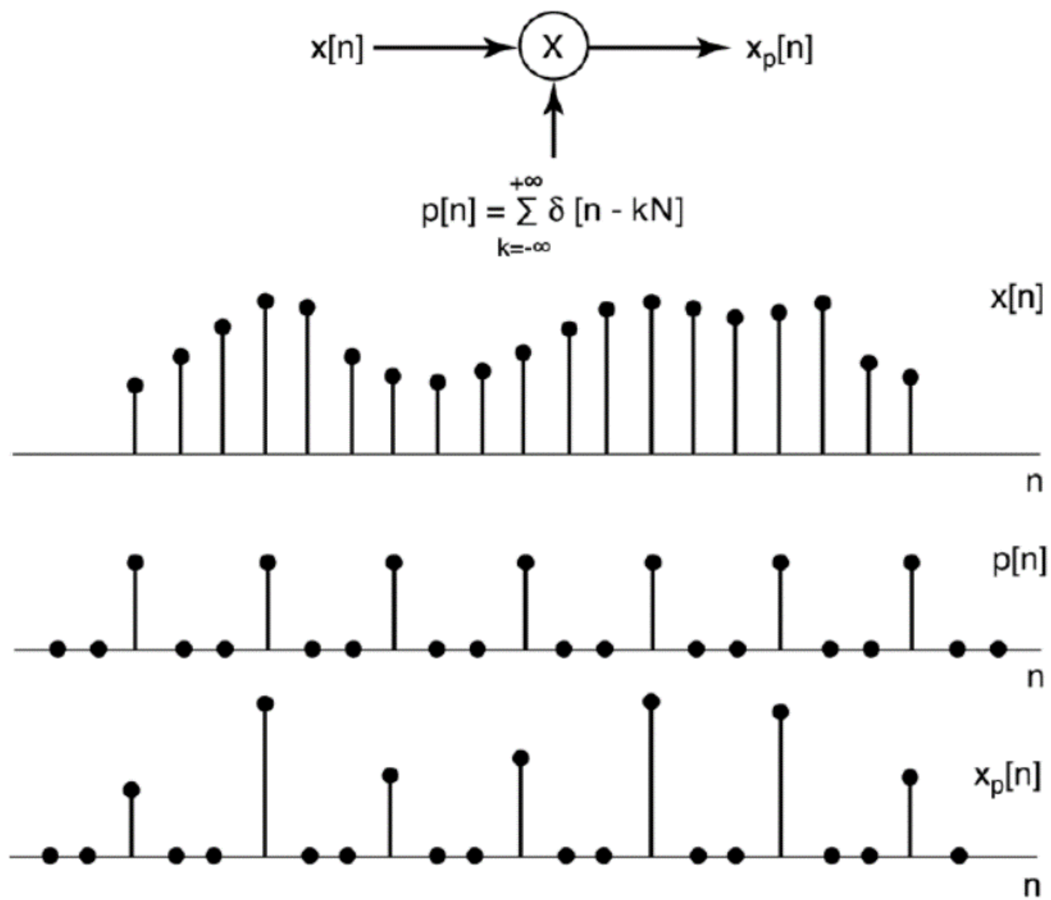
$$y[n] = \left(\frac{\sin(\pi n/4)}{\pi n}\right)^2 = x_1[n] \cdot x_2[n], \quad x_1[n] = x_2[n] = \frac{\sin(\pi n/4)}{\pi n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega})$$



2.4.2 DT Sampling

- Motivation: Reducing the number of data points to be stored or transmitted, e.g. in CD music recording.



$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \longleftrightarrow P(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s), \quad \omega_s = \frac{2\pi}{N}$$

$$\text{Note: } x_p[n] = x[n] \cdot p[n] = \sum_{k=-\infty}^{\infty} x[kN]\delta[n-kN]$$

$$x_p[n] = \begin{cases} x[n], & \text{if } n \text{ is integer multiple of } N \\ 0, & \text{otherwise} \end{cases} \Rightarrow \text{Pick one out of } N$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi} X(e^{j\omega}) \otimes P(e^{j\omega}) = \frac{1}{N} \sum_{k=0}^{N-1} X(e^{j(\omega - k\omega_s)})$$

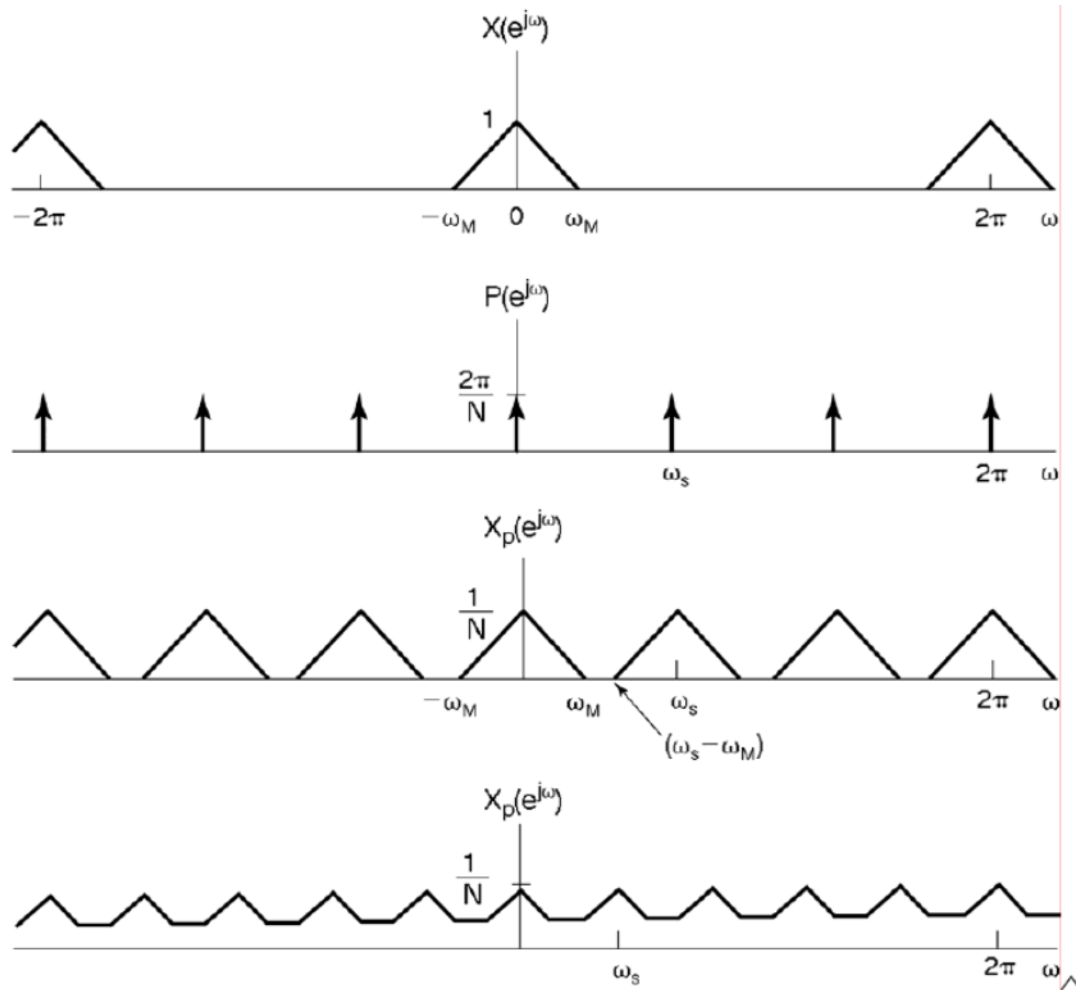
$$-\text{ periodic with period } \omega_s = \frac{2\pi}{N}$$

• DT Sampling Theorem :

We can reconstruct $x[n]$ if $\omega_s = 2\pi/N > 2\omega_M$

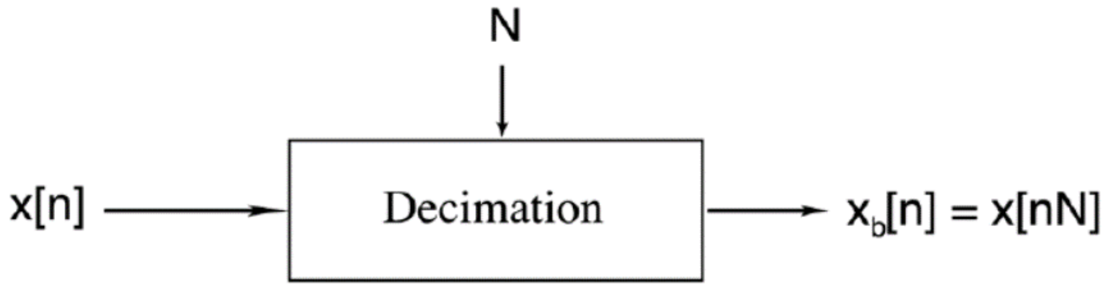
Drawn assuming $\omega_s > 2\omega_M$
Nyquist rate is met $\Rightarrow \omega_M < \pi/N$

Drawn assuming $\omega_s < 2\omega_M$
Aliasing!

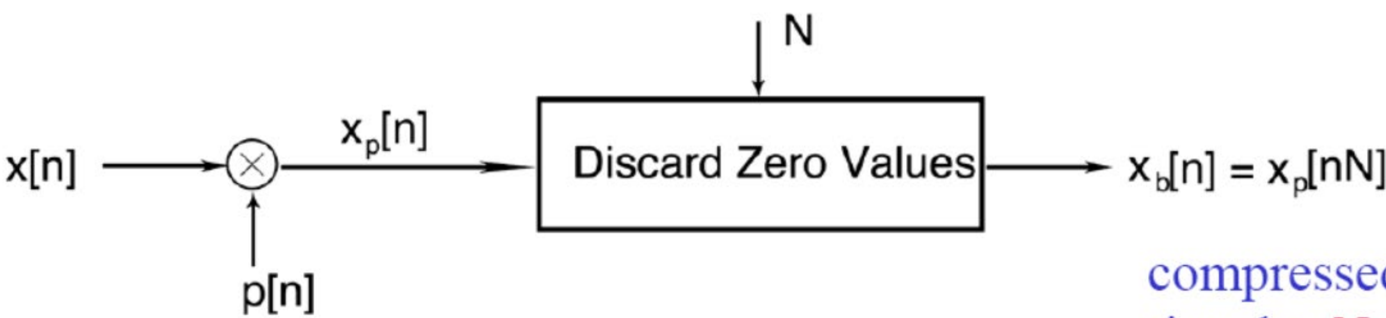


- Decimation — Downsampling:

$x_p[n]$ has (n-1) zero values between nonzero values: Why keep them around?

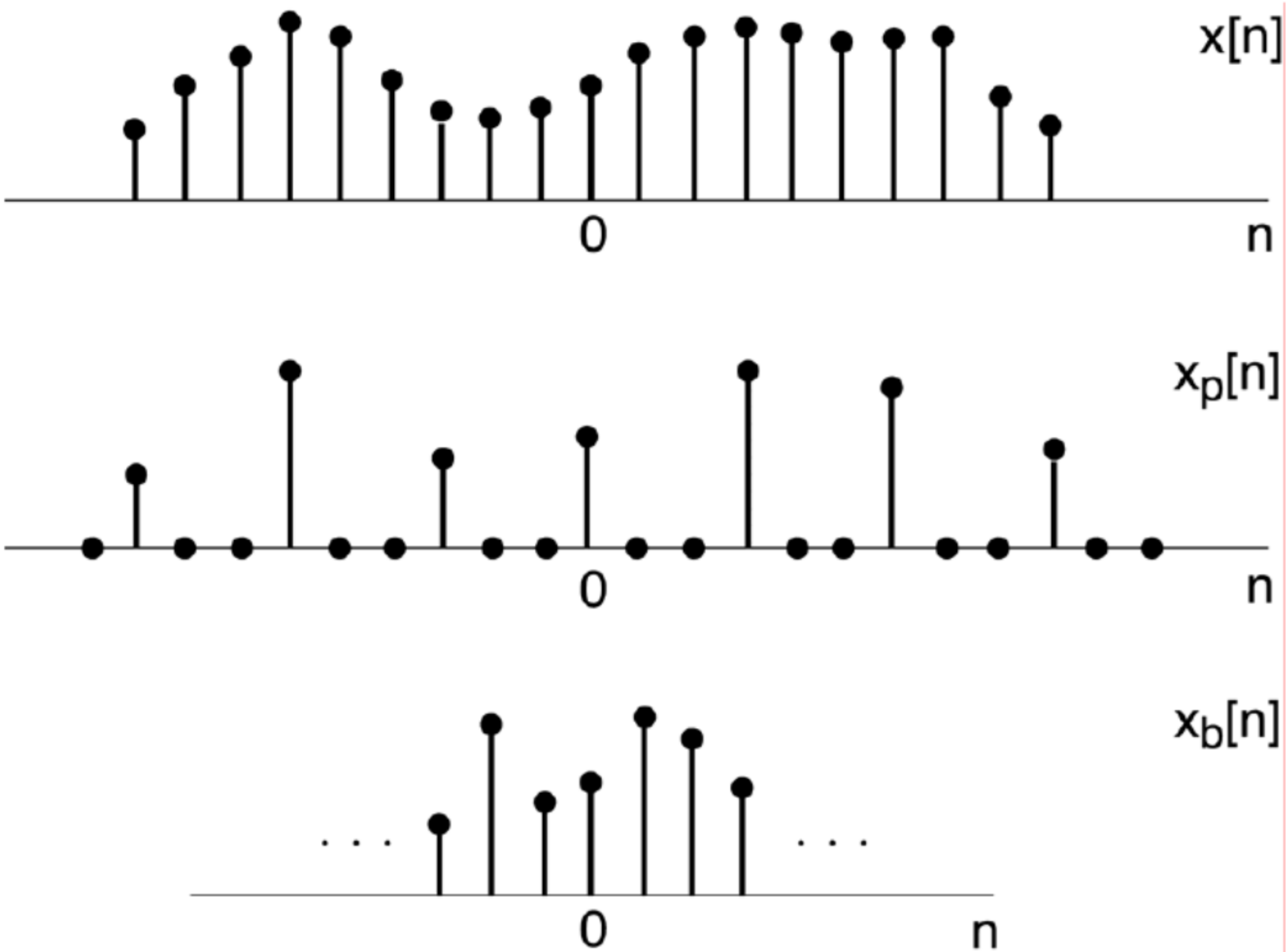


Useful to think of this as sampling followed by discarding the zero values



compressed in time by N

Illustration of Decimation in the Time-Domain (for N= 3)



Decimation in the Frequency Domain

$$X_b(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x_b[k] e^{-j\omega k} \quad (x_b[k] = x_p[kN])$$

$$= \sum_{k=-\infty}^{\infty} x_p[kN] e^{-j\omega k} \qquad \text{Let } n = kN \text{ or } k = n/N$$

$$= \sum_{\substack{n = \text{ an integer} \\ \text{multiple of } N}} x_p[n] e^{-j\omega(n/N)}$$

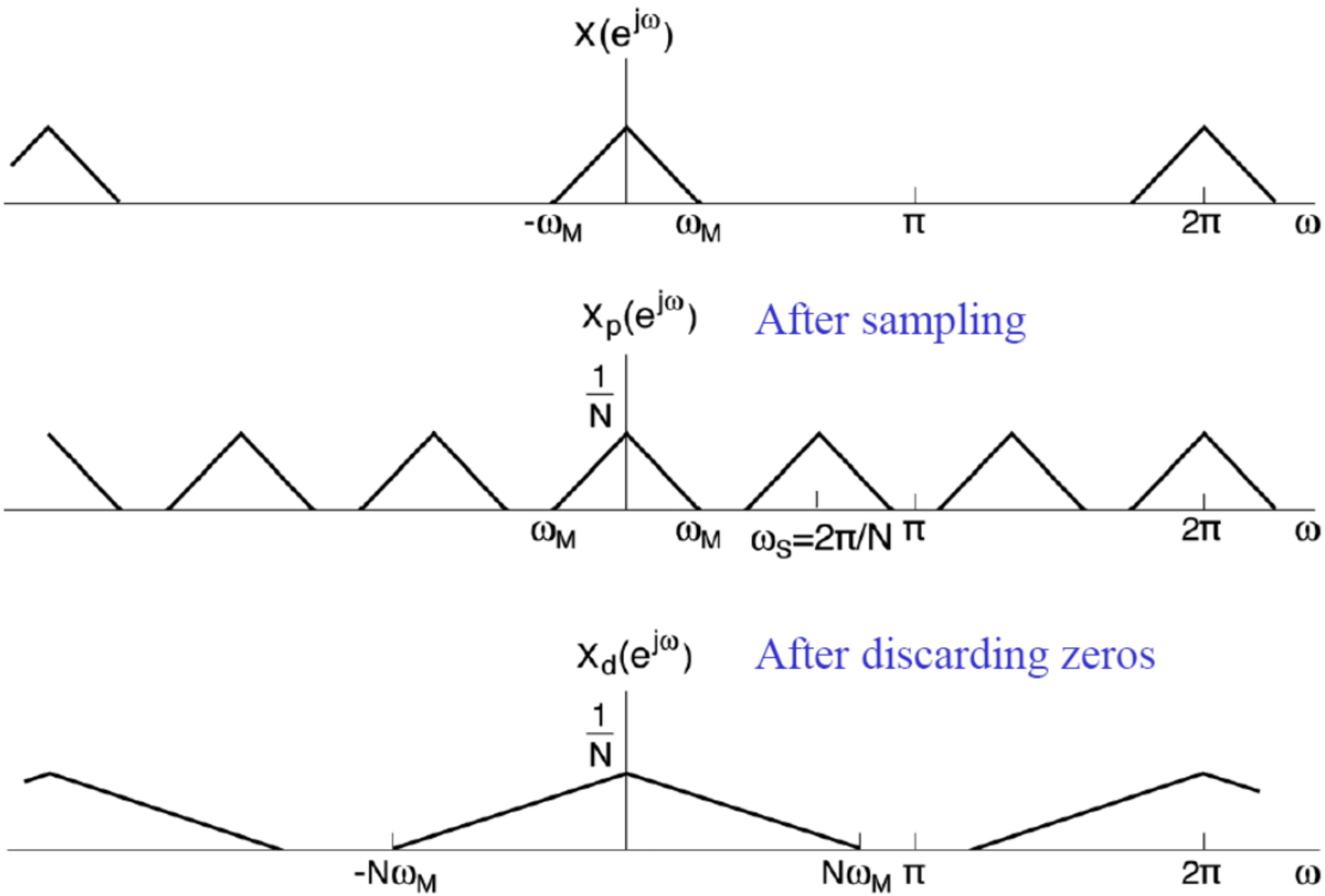
$$= \sum_{n=-\infty}^{\infty} x_p[n] e^{-j(\omega/N)n} \quad (\text{Since } x_p[n \neq kN] = 0)$$

$$= X_p(e^{j(\omega/N)})$$

Squeeze in time
Expand in frequency

- Still periodic with period 2π
  since $X_p(e^{j\omega})$ is periodic with $2\pi/N$

Illustration of Decimation in the Frequency Domain



Duality in Fourier Analysis

- CTFT: Both time and frequency are continuous and in general aperiodic

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Same except for these differences

Suppose *f*() and *g*() are two functions related by

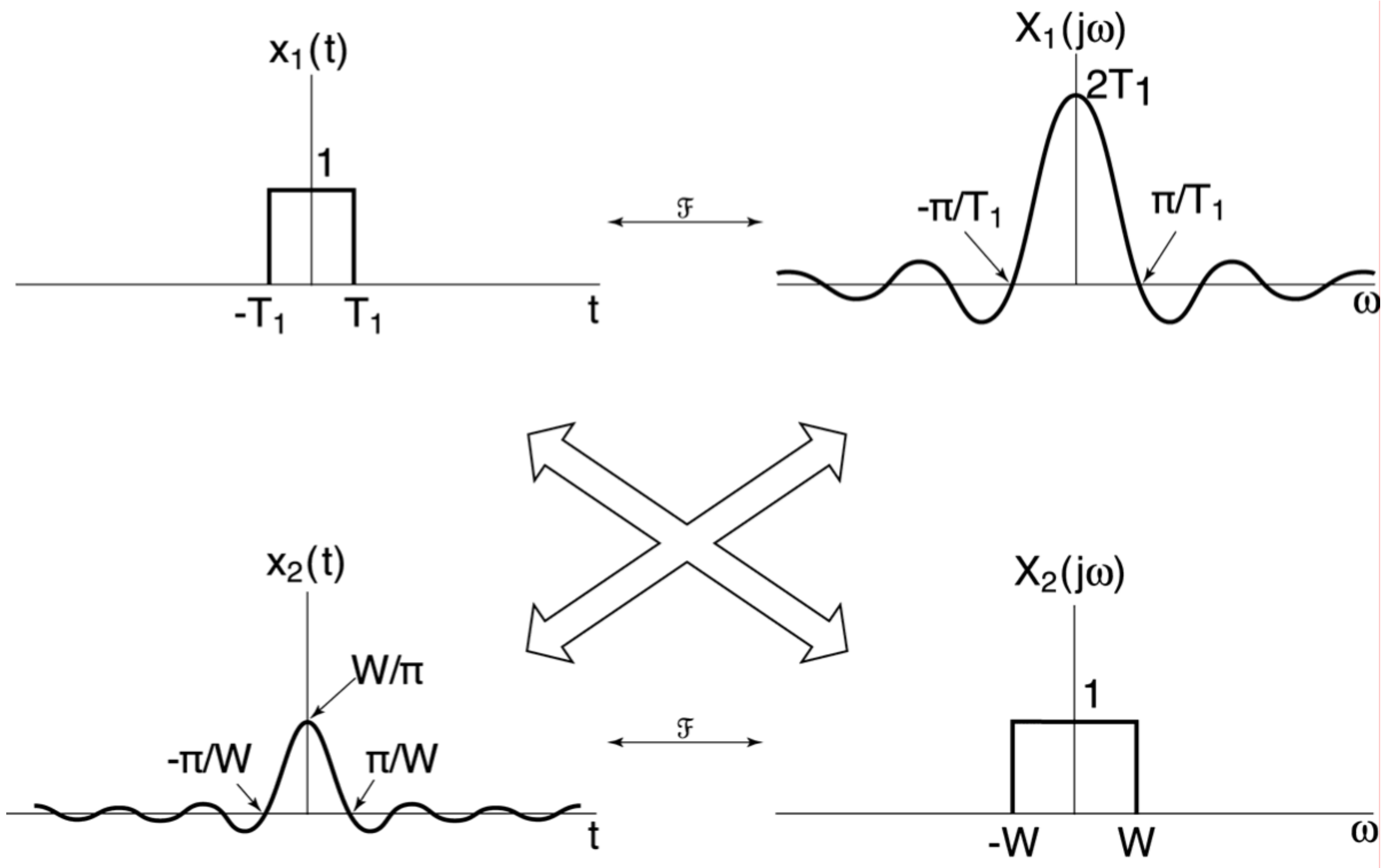
$$f(r) = \int_{-\infty}^{\infty} g(\tau)e^{-jr\tau}\,d\tau$$

Let $\tau = t$ and $r = \omega$: $\quad x_1(t) = g(t) \longleftrightarrow X_1(j\omega) = f(\omega)$

Let $\tau = -\omega$ and $r = t$: $\quad x_2(t) = f(t) \longleftrightarrow X_2(j\omega) = 2\pi g(-\omega)$

Example of CTFT duality

Square pulse in either time or frequency domain



Duality between CTFS and DTFT

CTFS

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = x(t+T), \quad \omega_0 = \frac{2\pi}{T}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

DTFT

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = X\left(e^{j(\omega+2\pi)}\right)$$

Q & A

Many Thanks